# Deterministic particles and smoothing effects[1]

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Work in collaboration with
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New Perspectives in Nonlocal and Nonlinear PDEs
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- $V[\rho_t](x) = v(\rho_t(x))$, $\rho_t$ absolutely continuous w.r.t. $\mathcal{L}^d$.
- $V[\rho_t](x) = -\nabla \rho_t^\gamma$ for some $\gamma > 0$, nonlinear diffusion, better regularity on $\rho_t$.

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$$\begin{cases} \dot{x}_i^N(t) = V^N(x_i^N(t)) \\ x_i^N(0) = x_{i,0}^N \end{cases} \qquad i = 1, \ldots, N\,. \tag{MP}$$

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or a similar discrete measure constructed out of the $N$ moving particles.
- Prove that $\mu_t^N \to \rho_t$ (in some sense), with $\rho_t$ solving (CE) with initial condition $\rho_0$.

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- Link with Lagrangian dynamics, justification of the PDE in discrete modelling setting.
- Numerical purpose: simulation via ODEs.
- Reduction to finite dimension, reduction of complexity: do we have a structure that is similar to the continuum model?
- Applications: in several context a particle simulation is more interesting than a PDE simulation (e.g. traffic flow).

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- Scalar conservation law
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with $f$ strictly convex or concave: Oleinik-Hoff estimate (1963-1983)
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which implies an $L^\infty$-$BV_{\mathrm{loc}}$ smoothing effect.
- Porous medium equation
$$\rho_t = \Delta \rho^\gamma \qquad \gamma > 1\,.$$
Fundamental estimate by Aronson and Bénilan (1979)
$$\Delta\left(\frac{m\,\rho^{m-1}}{m-1}\right) \geq -\frac{1}{(m+1)t}$$
which implies, among other things, an $L^1$-$L^\infty$ smoothing effect.

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Remarks:

- Controlling $R_i(t)$ from above is equivalent to controlling

$$d_i(t) = x_{i+1}(t) - x_i(t)$$

  from below. Estimates should be *uniform in $N$*.
- Clearly, we have some freedom in the choice of the deterministic approximation.
- In order to ensure consistency with these effects, local interactions should be rendered via *nearest neighbor interactions*.

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# Nonlocal repulsive equation

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but we prefer instead

$$\dot{x}_i(t) = -\frac{1}{N} \sum_{j=1}^{N} \frac{W(x_{j+1}(t) - x_i(t)) - W(x_j(t) - x_i(t))}{x_{j+1}(t) - x_j(t)} = -W' * \rho^N(t)(x_i(t))$$

where

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Integrating in time gives

$$d_i(t) \geq d_i(t)e^{-\frac{t}{2}} + \frac{2}{N}\left(1 - e^{-\frac{t}{2}}\right) \geq \frac{2}{N}\left(1 - e^{-\frac{t}{2}}\right)$$

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- The estimate is uniform in $N$ and does not involve the initial $L^\infty$ norm of $\rho^N$.

# A simple $L^1$ to $L^\infty$ smoothing effect

The above estimate reads for $R_i(t)$ as

$$R_i(t) = \frac{1}{Nd_i(t)} \leq \frac{2}{(1-e^{-t})} \qquad t > 0$$

and for $\rho^N(x,t)$ as

$$\|\rho^N(\cdot,t)\|_{L^\infty(\mathbb{R})} = \max_{i=1,\dots,N} R_i(t) \leq \frac{2}{(1-e^{-t})} \qquad t > 0$$

which is the desired smoothing effect.

- Instantaneous $L^\infty$ regularization, no matter whether the initial $L^\infty$ norm is finite or not.
- The above estimate is rigorous for particles that are initially detached.
- The estimate is uniform in $N$ and does not involve the initial $L^\infty$ norm of $\rho^N$.
- It may be extended to particles which are initially overlapping. The model can be formulated suitably as a finite dimensional gradient flow in which initially overlapping particles detach instantaneously.

# A nonlocal-to-local limit

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with probability measure initial data.
- Convergence of the many particle limit to gradient flow solutions.
- Scaled version of the particle system with potential
$$W_\varepsilon(x) = \varepsilon^{-1}W(\varepsilon^{-1}x)\,,$$
corresponding to
$$\partial_t\rho = \partial_x\left(\rho\partial_x W_\varepsilon * \rho\right)\,,$$
convergence to weak solutions to the quadratic PME
$$\partial_t\rho = \partial_x\left(\rho\partial_x\rho\right)$$
for initial data in $L^1 \cap L^\infty$.

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# Entropy solutions à-la Oleinik-Hoff for conservation laws

Scalar conservation law $\rho_t + f(\rho)_x = 0$ with initial datum $\overline{\rho} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

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- Oleinik-Hoff condition provides a *smoothing effect* from $L^\infty$ to $BV_{\mathrm{loc}}$ in space.
- The equality sign in Hoff's condition is achieved in case of *rarefaction waves* $f'(\rho(x,t)) = x/t$.

# DPA for scalar conservation laws

- Also known as *follow-the-leader* approximation, i.e. approximation of entropy solutions to

$$\partial_t \rho + \partial_x f(\rho) = 0\,, \qquad \rho(x,0) = \overline{\rho}(x) \geq 0$$

via the $(n+1)$-particle system

$$\dot{x}_i(t) = v\left(R_i(t)\right)\,, \qquad i = 0, \dots, N-1\,, \quad \dot{x}_N(t) = v(0)\,,$$

where we assume

$$\rho \mapsto v(\rho) = \frac{f(\rho)}{\rho} \text{ decreasing}\,, \qquad \rho \mapsto f(\rho) \text{ concave}\,,$$

through the piecewise constant reconstruction of the density

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$$\rho^N(x,t) = \sum_{i=0}^{N-1} R_i(t) \mathbf{1}_{[x_i(t), x_{i+1}(t))}(x)\,.$$
- It works (so far) only if $\overline{\rho} \geq 0$.
- It has a natural interpretation in traffic flow modelling.

# Main result in [DF-Rosini 2015]

- The discrete density

$$\rho^N(x,t) = \sum_{i=0}^{N-1} R_i(t)\mathbf{1}_{[x_i(t),x_{i+1}(t))}(x), \quad R_i(t) = \frac{1}{N(x_{i+1}(t) - x_i(t))}\,,$$

converges strongly in $L^1_{\mathrm{loc}}(\mathbb{R}\times[0,+\infty))$ as $N\to+\infty$ towards the unique entropy solution to the conservation law with $\overline{\rho}$ as initial condition.

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  - (i) in case the initial datum $\overline{\rho}$ has bounded variation,
  - (ii) in case the map $\rho \mapsto \rho v'(\rho)$ is non-increasing with no extra assumptions on $\overline{\rho}$.

# The entropy condition in the scheme

- In (ii), that is in case $\rho \mapsto \rho v'(\rho)$ is non-increasing, the estimate

$$\frac{v(\rho^N(x_{i+1}(t),t)) - v(\rho^N(x_i(t),t))}{x_{i+1}(t) - x_i(t)} \leq \frac{1}{t} \tag{3}$$

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- The above estimate is not sharp. In fact, in the continuum limit would read

$$v(\rho)_x \leq 1/t \tag{4}$$

  Recalling $f'(\rho) = v(\rho) + \rho v'(\rho)$, entropy solutions are characterised by Hoff's condition $f'(\rho)_x \leq 1/t$.

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- In [DF-Stivaletta, JHDE 2022] we prove

$$\frac{f'(\rho^n(x_{i+1}(t),t)) - f'(\rho^n(x_i(t),t))}{x_{i+1}(t) - x_i(t)} \leq \frac{1}{t} \tag{5}$$

  in the special case

$$v(\rho) = A - B\rho^\gamma\,, \qquad A \in \mathbb{R}\,, \quad B,\gamma > 0\,. \tag{6}$$

# A proof of the discrete Hoff estimate

Consider the case

$$v(\rho) = 1 - \rho\,.$$

The estimate reads in this case

$$D_i(t) = t\,\frac{R_i(t) - R_{i+1}(t)}{x_{i+1}(t) - x_i(t)} = tNR_i(R_i - R_{i+1}) \leq \frac{1}{2}\,, \qquad t > 0 \quad i = 0, \ldots, N-1\,.$$

Proof by backward induction: first prove the case $i = N-1$, then assume it is true for all $j > i$ and prove it for the $i$-th term.

$$\dot{D}_i = NR_i(R_i - R_{i+1}) + tN\dot{R}_i(R_i - R_{i+1}) + tNR_i(\dot{R}_i - \dot{R}_{i+1})$$

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$$\begin{aligned}
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=& NR_i(R_i - R_{i+1})\,[1 - D_i] + NR_iR_{i+1}{\color{red}D_{i+1}} - NR_i^2 D_i \\
& \text{we now use the inductive assumpion } {\color{red}D_{i+1}} \le 1/2 \\
\le& NR_i(R_i - R_{i+1})\,[1 - D_i] + \frac{1}{2}NR_iR_{i+1} - NR_i^2 D_i {\color{blue}\pm NR_iR_{i+1}D_i}
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# A proof of the discrete Hoff estimate

Consider now the positive part $(x)_+ = \max\{0, x\}$ and its derivative $H(x)$ the Heaviside function. We get

$$\begin{aligned} &\frac{d}{dt}(D_i(t))_+ = H(D_i)\dot{D}_i \\ =&H(D_i)NR_i\left(R_i - \frac{1}{2}R_{i+1}\right)(1-2D_i)\,. \end{aligned}$$

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Since $D_i \geq 0$ if and only if $R_i \geq R_{i+1}$, we get

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Hence, we obtain

$$D_i(t) \leq \frac{1}{2}$$

via a simple ODE comparison argument.

# Table of contents

# Nearest neighbor interaction scheme for the PME[2]

One dimensional PME reads

$$\rho_t - \left(\rho\left(\frac{m}{m-1}\rho^{m-1}\right)_x\right)_x = 0\,.$$

[2]Work in preparation with D. Matthes (TUM)

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$$\mathcal{P}(\mathbb{R}) \ni \rho(\cdot,t) \mapsto X(\cdot,t) \in \left\{X \in L^2((0,1)) \text{ non-decreasing}\right\}$$
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\end{aligned}$$

Velocity field in the continuity equation PME

$$\begin{aligned}
\dot{x} &= -\left(\frac{m}{m-1}\rho^{m-1}\right)_x = -\left(\partial_z X\right)^{-1}\left(\frac{m}{m-1}\rho^{m-1}\right)_z\\
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To construct our scheme, we turn the last derivative above into a finite difference.

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We define the particle scheme

$$\begin{cases} \dot{x}_i(t) = -N(R_i(t)^m - R_{i-1}(t)^m) \quad i = 1, \ldots, N-1 \\ \dot{x}_0(t) = -NR_0(t)^m \\ \dot{x}_N(t) = NR_{N-1}(t)^m \, . \end{cases}$$

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We derive the ODE system for the densities $R_i$:

$$\dot{R}_i(t) = \begin{cases} N^2 R_0(t)^2 (R_1^m(t) - 2R_0^m(t)) & \text{if } i = 0 \\ N^2 R_i(t)^2 \left(R_{i+1}^m(t) + R_{i-1}^m(t) - 2R_i^m(t)\right) & \text{if } i \in \{1, \dots, N-2\} \\ N^2 R_{N-1}(t)^2 (R_{N-2}^m(t) - 2R_{N-1}^m(t)) & \text{if } i = N-1 \end{cases}$$

# Nearest neighbor interaction scheme for the PME

We define the particle scheme

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which may be re-written as

$$\begin{aligned} &\dot{R}_i(t) = R_i(t)^2 (\Delta R(t)^m)_i \\ &(\Delta R(t)^m)_i = N^2 (R_{i+1}(t)^m + R_{i-1}(t)^m - 2R_i(t)^m) \\ &\text{with the convention } R_{-1}(t) = R_N(t) = 0 \, . \end{aligned}$$

# Discrete Aronson-Bénilan estimate

Aronson Bénilan estinate $\left(\dfrac{m}{m-1}\rho^{m-1}\right)_{xx} \geq -\dfrac{1}{(m+1)t}$

In the pseudo-inverse variable $\rho\left(\rho^{m}\right)_{zz} \geq -\dfrac{1}{(m+1)t}\,.$

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$$\begin{aligned}\dot{Z}_i &= \dot{R}_i\,(\Delta R^m)_i + N^2 mR_i\big(R_{i+1}^{m-1}\dot{R}_{i+1} + R_{i-1}^{m-1}\dot{R}_{i-1} - 2R_i^{m-1}\dot{R}_i\big)\\ &= Z_i^2 + N^2 mR_i\big(R_{i+1}^{m}Z_{i+1} + R_{i-1}^{m}Z_{i-1} - 2R_i^{m}Z_i\big)\,,\end{aligned}$$

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Assume $Z_i(t)$ attains its minimal value at $i = k^*$ on the interval $[t, t+\varepsilon)$. Then,

$$\begin{aligned}\dot{Z}_{k^*}(t) &\geq Z_{k^*}(t)^2 + N^2 m R_{k^*}(t)\left(R_{k^*+1}(t)^m + R_{k^*-1}(t)^m - 2R_{k^*}(t)^m\right)Z_{k^*}(t)\\ &= Z_{k^*}(t)^2 + mR_{k^*}(t)\left(\Delta_{k^*}R(t)^m\right)Z_{k^*}(t)\\ &= (m+1)Z_{k^*}(t)^2.\end{aligned}$$

# Discrete Aronson-Bénilan estimate

Integrating in time we get

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### Application to the speed of propagation of the support

The above implies

$$\dot{R}_i(t) \geq -\frac{R_i(t)}{C+(m+1)t}$$

which gives

$$R_i(t) \geq R_i(0)(1+C^{-1}(m+1)t)^{-\frac{1}{m+1}}$$

and consequently

$$x_N(t)-x_0(t) = \sum_{i=0}^{N-1} d_i(t) = \frac{1}{N}\sum_{i=0}^{N-1} R_i(t)^{-1} \leq \sum_{i=0}^{N-1} d_i(0)(1+C^{-1}(m+1)t)^{\frac{1}{m+1}}$$
$$= (x_N(0)-x_0(0))(1+C^{-1}(m+1)t)^{\frac{1}{m+1}}\,.$$

# Table of contents

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where $W$ is the nonlocal operator

$$W[\rho](x) = v\left((V * \rho)(x)\right) = v\left(\int_{\mathbb{R}} V(x-y)\rho(y)dy\right)$$

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- A possible direction to investigate: fully discrete DPA. Work in preparation with Fagioli, Iorio and Rosini for scalar conservation laws.
- Huge downside: this talk is *sadly one-dimensional*. It would be nice to see similar properties in multi dimensional particle models.

# End of the talk

Thank you!